P''', respectively, giving us, therefore, two inscribed triangles, PQR and $P''Q''R''$, and two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$. We will now show that O is the common co-polar point of all four triangles, and HJI, along its polar, their three common collinear co-axial points.

For, by construction, O is the co-polar point of the inscribed triangles PQR and $P''Q''R''$. And if their sides PR and $P''R''$ cut in, say h, while QR and $Q''R''$ cut in, say i, and PQ, $P''Q''$ cut in, say j; then will O and h be summits of quadrangle $PRR''P''$, and likewise Oi summits of quadrangle $QRR''Q''$, and Oj summits of quadrangle $PQQ''P''$. Hence hji is the polar of O with respect to the conic; thus coinciding with line HJI. But the three points where the polar of O, the co-axial line of triangles PQR and $P'Q'R$, cuts the sides of triangle PQR, are HJI. So that lines HJI and hji not only coincide; but points Hh, Jj, and Ii, respectively, are identical; so that HJI are the common collinear co-axial points of the three co-polar triangles PQR, $P'Q'R'$, and $P''Q''R''$; their corresponding sides PR, $P'R'$, and $P''R''$ concurring in H; sides PQ, $P'Q'$, and $P''Q''$ concuring in J, and QR, $Q'R'$, and $Q''R''$ in I.

Then lastly, H being a summit of the quadrangle $PJIR$, must be the pole to the polar $QQ'Q''O$ joining its other two summits Q' and Q, and hence in it meet the two tangents $QP'HR'$ and $R'''Q''P'''H$, while the quadrangle $PJIR$ again gives us $RQ'JP'$ as a harmonic range. And thus $PQ'I$ and $OPP'P''$ as conjugate rays to the pencil $PQ'I$, PJQ, $PP'P''$, and RPH. So that in I concur the tangents $R'PQ'I$ and $P'Q'''IR'''$. While similarly, $RQ'JP'$ and $RR''R'$ being conjugate rays in the pencil RQ, RP', RH, and RR', it follows that in J concur the tangents $RQ'J$, and $R''P'''JQ'''$.

So that HJI are the three collinear points in which the two inscribed triangles PQR and $P''Q''R''$, and the two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$ are commonly co-axial, while by Theorem 1, triangles $P'Q''R''$ and $P'''Q'''R'''$, being so co-axial in HJI, must be co-polar in O, its pole, and thus all four triangles be commonly co-polar in said point O. So that $OPP'''P'P''$ are collinear; as also $OQ''Q'Q'''Q$, and $R'''ROR''R'$.

NOTE ON THE QUARTIC.

By DR. R. P. STEPHENS, Wesleyan University, Middletown, Conn.

The general quartic equation

$$ax^4+4bx^3+6cx^2+4dx+e=0, \tag{1}$$

where the coefficients may be real or complex, can be reduced to the form

$$x^4-2\mu x^2+\nu^2=0. \tag{2}$$

The roots of (2) are

$$x=\pm\sqrt{[\mu\pm\sqrt{(\mu^2-\nu^2)}]} \text{ or } x=\pm\frac{1}{\sqrt{2}}[\sqrt{(\mu-\nu)}\pm\sqrt{(\mu+\nu)}].$$

If we make the substitution

$$\sqrt{2}\,t\equiv\sqrt{(\mu-\nu)}\pm(\mu+\nu),$$

then the four roots may be expressed as

$$x_1=t,\quad x_2=-t,\quad x_3=\nu/t,\quad x_4=-\nu/t.$$

That any quartic which has no repeated roots can be reduced to the form (2) is easily seen geometrically. In general, the roots of a quartic are represented by four points in the complex plane, say x_1, x_2, x_3, x_4. If these be divided into pairs x_1, x_2 and x_3, x_4, there is a pair α, β which is harmonic* with respect to both. That bilinear transformation which throws α into zero and β into infinity will obviously throw x_1 and x_2 into points symmetrical with respect to the origin, and will arrange x_3 and x_4 in a similar manner; and the transformed points will be as given above.

The reduction of the quartic to this simple form is secured analytically as follows: If x_1, x_2, x_3, x_4 are the four roots of (1), then the pair α, β will be harmonic with respect to the two pairs x_1, x_2 and x_3, x_4, provided

$$2\alpha\beta-(\alpha+\beta)(x_1+x_2)+2x_1x_2=0,$$
$$\text{and } 2\alpha\beta-(\alpha+\beta)(x_3+x_4)+2x_3x_4=0;$$

from which

$$\alpha+\beta=\frac{2(x_1x_2-x_3x_4)}{x_1x_2(x_3+x_4)-x_3x_4(x_1+x_2)}.$$

But since the four roots may be grouped in two other ways, we shall have two other values for $(\alpha+\beta)$. Forming the equation of which these three values of $\alpha+\beta$ are roots and substituting the values for symmetric functions of the roots from (1), we obtain†

$$(3)\qquad 2(a^2d-3abc+2b^3)\lambda^3+(a^2e+6b^2c-9ac^2+2abd)\lambda^2$$
$$+2(abc+2b^2d-3acd)\lambda+b^2e-ad^2=0,$$

where $\lambda\equiv\alpha+\beta$.

In a similar way, from the reciprocals of the roots, we obtain

$$(4)\qquad 2(be^2-3cde+2d^3)\lambda'^3+(ae^2+6cd^2-9c^2e+2bde)\lambda'^2$$
$$+2(ade+2bd^2-2bce)\lambda'+ad^2-b^2e=0,$$

*For this general use of harmonic pairs, see Harkness and Morley, *Introduction to Analytic Functions*, pp. 32—34.

†See Burnside and Panton, *Theory of Equations*, p. 130 (3rd Ed.).

where $\lambda' \equiv 1/\alpha + 1/\beta$. From λ and λ', we can obtain α and β.

If now α_1, β_1 are a harmonic pair obtained from (3) and (4), then the bilinear transformation

$$x = \frac{\beta_1 x' + \alpha_1}{x' + 1}$$

will reduce the general quartic (1) to the required form.

In case two of the roots of (1) are equal, say $x_1 = x_2$, then equation (3) will have a repeated root which will equal twice the reciprocal of x_1, the repeated root of (1); and if three roots of (1) are equal, then all harmonic pairs coincide at the repeated root and (3) will be a perfect cube. Thus we see that the general quartic with real or complex coefficients can be reduced to the form (2) in this way.

REMARKS ON THE REPORT ON GEOMETRY OF THE COMMITTEE* OF THE CENTRAL ASSOCIATION.

By PROFESSOR GEORGE BRUCE HALSTED, Greeley, Colorado.

This report is epoch-making,—at the very least epoch-marking.

There simply *must* be things not *explicitly* defined, and point, straight, plane, between, should be among them. The Report says: "It is recommended that the term *sect* be used for segment of a straight line lying between two of its points." In fact the term "sect" has 'arrived.' The Encyclopaedia Americana uses it. This Report uses it more than twenty-two times. It is used not less than twenty-four times in the remarkable Presidential Address by Professor Alfred Baker of the University of Toronto to the Royal Society of Canada.

Think of the neatness with which the bunglesome phrase "transferrer of straight-line segments" becomes *sect-carrier*. Realize how elegantly "the algebra or algorithm of straight-line segments" becomes *sect-calculus*.

"Instead of *axioms*," says the Report, "use geometrical *assumptions*." It gives for example Pasch's assumption, now so renowned. From the list of assumptions we mention: "3. A point on a straight line divides it into two parts, called *rays*." "6. A sect has one and only one mid point." "11. A straight line divides the points not on it into two classes such that sects determined by two points of the same class are not intersected by the line, and sects determined by two points not of the same class are intersected by the line."

*The Committee consist of G. W. Greenwood, Chairman, Salem, Va.; C. A. Pettersen, Chicago, Ill.; C. E. Comstock, Peoria, Ill., and C. W. Newhall, Faribault, Minn. Copies of the report may be had by sending a stamp to Miss Mabel Syker, 438 East 57th Street, Chicago, Ill. Ed. S.

"Definitions," says the Report, "should not be based upon crude images, affording little upon which reasoning may lay hold. For example, 'An angle is the opening between two lines which meet.' Instead, define an angle as the figure formed by two rays having a common origin." A demonstration in which we use information obtained by looking at a figure is not of the highest order.

The one serious slip in the Report is the sentence: "Also, in 'A line perpendicular to each of two intersecting lines (at their intersection) is perpendicular to their plane,' we assume that two intersecting lines have a common perpendicular though we cannot justify the assumption by any previous proposition." Halsted's Rational Geometry here makes no assumption whatever. Its figure for this proposition is already covered by the preceding proposition: On any straight to put two planes; and the problem: To erect a perpendicular to a straight from any point on it.

We must agree with the Report, that the treatment of mensuration in most texts is extremely unfortunate. In fact measurement in terms of a common unit at once introduces incommensurability and irrational numbers. No geometry exists in which irrational numbers are adequately treated. Halsted's Rational Geometry outwits the difficulty.

The committee recommends that a critical course in elementary geometry be offered in courses of study in colleges.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Remarks on Two Solutions of Problem 89. By G. A. MILLER.

The following solutions present very instructive examples of fallacious reasoning and arriving at the answer by a remarkable coincidence. As the problem is so well known, and these solutions are said to have appeared in other scientific journals it seems desirable to enter into some details. The problem is as follows:

Solve by quadratics, $x^2+y=7...(1)$, $x+y^2=11...(2)$.

We shall first speak of the solution given on page 37, Volume VI, of this journal. To make the matter as clear as possible we shall employ the language of analytic geometry. The problem is to find the common points (or at least one of them) of two intersecting parabolas. The author of the solution in question subtracts (2) from (1), and thus obtains the equation of an equilateral hyperbola containing the four common points of the given parabolas. The equation of this hyperbola is

$$y^2-x^2-(y-x)=4\ldots(3).$$

Substituting a for $x+y$ and b for $y-x$ tnis equation reduces to

$$ab-b=4\ldots(4).$$

While hyperbola (4) is satisfied by an infinite number of pairs of values for a and b, the author arrives at the pair $a=5$, $b=1$ by taking the following steps.

From (4) we may obtain by transposing and squaring,

$$a^2b^2=16+8b+b^2\ldots(5),$$

and also
$$-10ab+25=-15-10b\ldots(6).$$

Adding (5) and (6) there results

$$(ab-5)^2=(1-b)^2,$$

or
$$ab-5=1-b,\ ab+b=6\ldots(7).$$

Combining (4) and (7), it follows that $b=1$ and $a=5$.

The author has thus found one point on hyperbola (4), and the remarkable coincidence is that this is a point of intersection of the parabolas (1) and (2). Hence $x+y=5$, $y-x=1$ lead to $x=2$, $y=3$, which is a solution of the equations. That the method is erroneous follows directly from the fact that the same auxiliary equations could be obtained from

$$x^2+y=k\ldots(1),$$
$$x+y^2=k+4\ldots(2),$$

where k is arbitrary. If we make $k=0$, for instance, it is evident that $x=2$, $y=3$ does not satisfy the system. The solution is a good illustration of blindly manipulating algebraic expressions. Equations (3) and (4) represent a hyperbola which goes through the points whose co-ordinates are desired, and the rest of the solution is merely a very laborious method of finding the co-ordinates of one point on this hyperbola. As the number of points on the hyperbola is infinite, while only four of these points are common to the given parabolas, the probability that this method should lead to a correct result in a given problem is zero, and yet it happened to do so in the present instance.

The second solution of the same problem, to which we desire to call attention, is published on page 13 of the same volume. Equations (1) and (2) are written as follows:

$$x^2-9=2-y=d, \text{ by assumption,}$$
$$x-3=4-y^2=sd, \text{ by assumption.}$$

Hence
$$x^2-9=(x-3)/s=x/s-3/s.$$

Completing the square we have

$$x^2-x/3+1/4s^2=9-3/s+1/4s^2.$$

Hence $x-1/2s=3-1/2s$, or $x=3$.

The fallacy will at once appear if it is observed that the same arguments could be used with respect to the two equations

$$x^2+y=9+\alpha\ldots(1),$$
$$y^2+x=3+\beta\ldots(2).$$

These may be written as follows:

$$x^2-9=\alpha-y=d\ldots(3),$$
$$x-3=\beta-y^2=sd\ldots(4).$$

As x cannot be equal to 3 for an arbitrary pair of values of α, β it follows that the method is fallacious. Just as in the preceding solution it is implicitly assumed that any point on $x^2-9=(x-2)/s$ must be common to the given parabolas. And the probability that this method should lead to the correct result in a given problem is again zero.

278. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

$xyz(\Sigma x)^2<3\Sigma y^2z\,\Sigma yz^2$, if x, y, z are positive.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{x^2y+y^2z+z^2x}{3}>(x^3y.y^3z.z^3x)^{1/6}>xyz. \quad \text{Also } \frac{xy^2+yz^2+zx^2}{3}>xyz.$$

$\therefore 3\,\Sigma y^2z\,\Sigma yz^2>27x^3y^3z^3>xyz.27xyz.$

$(x+y+z)^2$ is the greatest when $x=y=z$, and is then equal to $9x^2$.

$\therefore (x+y+z)^2<27xyz. \quad \therefore xyz[\Sigma(x)]^2<3\,\Sigma y^2z\,\Sigma yz^2.$

279. Proposed by THEODORE L. DE LAND, Treasury Department, Washington, D. C.

The United States Panama Canal Bonds were issued, to date August 1, 1906, and will mature on August 1, 1936; and they bear interest at the rate of 2% per annum, payable quarterly, on the first day of November, 1906, and the first day of February, May, and August, 1907, and so on for each succeeding quarter, until the bonds mature, when the principal will be paid at par with the last quarter's interest. The coupon bonds of this loan were quoted on the New York Stock Exchange, at 10.30 a. m., on December 17, 1906, at $103\frac{3}{4}$ bid and $104\frac{1}{4}$ asked.

Required: The rate of interest per annum, payable quarterly, an investor would *realize* if he purchased the Panama bonds on December 17, 1906, and could reinvest his interest income, quarterly, at the *realized* rate.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let P=price of bond, n=number of quarter years to run, S=face of bond, x=realized rate of interest, r=rate of interest bond bears.

$P(1+\frac{1}{4}x)^n$=value of purchase money at the end of n quarter years.

$$\frac{Sr}{4}(1+\tfrac{1}{4}x)^{n-1}+\frac{Sr}{4}(1+\tfrac{1}{4}x)^{n-2}+\frac{Sr}{4}(1+\tfrac{1}{4}x)^{n-3}+\ldots+Sr+S$$

$$=S+\frac{\frac{Sr}{4}[(1+\frac{1}{4}x)^n-1]}{\frac{1}{4}x}$$

is the amount of money received on the bond.

$$\therefore P\tfrac{1}{4}x(1+\tfrac{1}{4}x)^n=\frac{Sx}{4}+\frac{Sr}{4}(1+\tfrac{1}{4}x)^n-\frac{Sr}{4}.$$

$$\therefore 104\tfrac{3}{4}(1+\tfrac{1}{4}x)^{119}x=100x+2(1+\tfrac{1}{4}x)^{119}-2,$$

$$(104\tfrac{3}{4}x-2)(1+\tfrac{1}{4}x)^{119}=100x-2,$$

$$\therefore x=.01794=1.794\%.$$

II. Solution by the PROPOSER.

The solution of the above problem requires that certain technical considerations be observed. The bond runs for 30 years, and as the interest is payable quarterly there are attached to it 120 coupons of \$0.50 each. The exact number of days in each quarter must be noted. The interest for the first quarter was paid November first. The next quarter (November, December, and January), contains 92 days, and interest has accrued from November first to include December sixteenth, or for 46 days; or the accrued interest is equal to $\frac{46}{92}$ of a quarter; or is equal to $\frac{1}{2}$ of \$0.50, or \$0.25, to the date of purchase. There is left to the date of the maturity 120−1.5 or 118.5 quarters. The lowest price bid was \$103.75, and highest price asked was \$104.75. If there was a sale of these bonds on December 17 it is fair to assume that it was at either limit or between those limits, and we therefore assume that the sale was at the mean price or at \$104.25. To find the net investment we subtract the accrued interest from the mean price and have the net investment, or \$104.25−\$0.25, or \$104, the net investment.

Let $4X$=the realized rate of interest per annum, payable quarterly; or let X=the quarterly rate; let P=\$100=the face of the bond; n=the number of interest periods; I=the quarterly interest on \$100; and N=\$104=the net investment.

The amount of the net investment from the date of purchase to the date of maturity would be expressed symbolically by $N(1+X)^n$. The investor will at the date of maturity receive the face of the bond, P=\$100, and the amount of his quarterly interest considered as an annuity compounded at the rate X for the time n, which would be expressed, symbolically, by $I[(1+X)^n-1]/X$.

We can now equate the equivalent terms and obtain the following general equation:

$$N(1+X)^n=P+I[(1+X)^n-1]/X \ldots (1).$$

By transformation and reduction (1) takes the following general form:

$$N=I/X-(I/X-P)/(1+X)^n \ldots (2).$$

Substituting numerical values we have:

$$\$104=\$0.50/X-[\$0.50/X-\$100]/(1+X)^{118.5} \ldots (3).$$

We now have to find such a value of X that if it be substituted in (3) the two members will be equal. As the bond is sold at a premium we know that X must be less than 0.005. By trial we find it to be greater than 0.0045. We will first try 0.00456, and then 0.00457.

Take $X=0.00456$ in (2) and (3) and we have $I/X=\$0.50/0.00456=\$109.649123$; from this amount take \$100 and we have \$9.649123. The logarithm of 9.649123 is 0.9844878. From this logarithm take 118.5 times the logarithm of 1.00456, or 0.2341442, and we have the logarithm 0.7503436; which is the logarithm of the number 5.627864; this number taken from 109.649123 gives 104.021259. This number is greater than the number in the first member of equation (3), or than 104, by 0.021259, which shows that the assumed value of X, or 0.00456, is too small.

Take $X=0.00457$ in (2) and (3) and we have $I/X=\$0.50/0.00457=\$109.40919$; from this amount take \$100, and we have \$9.40919. The logarithm of 9.40919 is 0.9735522; from this logarithm take 118.5 times the logarithm of 1.00457, or 0.2346537, and we have the logarithm 0.7388985, which is the logarithm of the number 5.47149; this number taken from the number 109.40919 gives 103.93770. This number is less than the number in the first member of equation (3), or than 104, by 0.0623, which shows that the assumed value of X, or 0.00457, is too large.

Now as the assumed value, $X=0.00456$, gives a value too small, and the assumed value, $X=0.00457$, gives a value too large, the true value of X lies between these two assumed rates, the first assumption giving the nearer value. The approximate increase may be determined by Double Position:

104.02126	0.00457	·	104.021259	
103.93770	0.00456		104.000000	
0.08556	: 0.00001	::	0.021259	: error=e.

Reducing the proportion we find $e=0.00000254$; add this to 0.00456 and we have 0.00456254 as a new trial value for X.

Try $X=0.0045625$ in equation (3) and we have a result approximately within a few mills of a true value.

Therefore the rate of interest per quarter which the investor will realize is $X=0.45625\%$; or $4X=1.825\%$, the investor's rate per annum, payable quarterly.

When there are three rates of interest to be considered in the problem, or the nominal rate expressed in the bond, the investor's rate, and the current or market rate, each per annum, payable quarterly, we let $4Y$=the current rate per annum, payable quarterly, then Y=the current quarterly rate. Put this value in equation (1) and we have:

$$N(1+X)^n=P+I[(1+Y)^n-1]/Y\ldots(4).$$

With equation (4) we have a perfectly general equation from which, by proper substitutions, we may apply it to any loan and find all values required, as N, P, I, n, X, and Y.

The logarithmic calculations in the above solution were made with Shortrede's tables of logarithms and antilogarithms. Should it be necessary to carry the decimals further it would be better to use ten-place tables.

Dr. Zerr's result differs from Mr. DeLand's in that he disregarded the technical consideration required in these bonds. The problem was also solved by A. H. Holmes. ED. F.

280 (Incorrectly numbered 180). Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find values of x, y, z, and u satisfying the equations

$$x+y+z+u=10\ldots[1],$$
$$x^2+y^2+z^2+u^2=30\ldots[2],$$
$$x^3+y^3+z^3+u^3=100\ldots[3],$$
$$x^4+y^4+z^4+u^4=354\ldots[4].$$

Solution by E. A. ECKHARDT, 903 North Fifth Street, Philadelphia, Pa.

By multiplication, addition, and subtraction we obtain

$\Sigma xy=35$
$\Sigma x^2y=300$
$\Sigma xyz=50$
$\Sigma xy^3=646$
$\Sigma x^2y^2=273$
$\Sigma x^2yz=404$
$xyzu=24$

$$\begin{aligned}\Sigma xy^3&=x^3y+x^3z+x^3u+y^3x+y^3z+y^3u+z^3x+z^3y\\&\quad+z^3u+u^3x+u^3y+u^3z.\\&=x^2(xy+xz+xu+yz+yu+zu)\\&\quad-x(xzu+xyu+xyz+yzu)+xyzu\\&\quad+y^3(x+u+z)+z^3(x+y+u)+u^3(x+y+z)\\&=x^2\Sigma xy-4\Sigma xyz+xyzu+y^3(10-y)\\&\quad+z^3(10-z)+u^3(10-u).\end{aligned}$$

$$\Sigma xy^3=x^2\Sigma xy-4\Sigma xyz+xyzu+10(y^3+z^3+u^3)-(y^4+z^4+u^4).$$

Substituting, we have $646=35x^2-50x+24+10(100-x^3)-(354-x^4)$.

Whence, we have $x^4-10x^3+35x^2-50x=-24\ldots(2)$,

or $x^4-10x^3+35x^2-50x+24=0$.

The equations in y, z, and u are found to be identical to (2). The

roots of (2) are easily found to be $x=1$, $x=2$, $x=3$, $x=4$. We have, therefore, the following four groups of values:

$$\begin{cases} x=1,\ y=2,\ z=3,\ u=4; \\ x=2,\ y=1,\ z=4,\ u=3; \\ x=3,\ y=4,\ z=1,\ u=2; \\ x=4,\ y=3,\ z=2,\ u=1. \end{cases}$$

Also solved by G. B. M. Zerr, J. Scheffer, and A. H. Holmes.

GEOMETRY.

308. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the locus of O, if the differences of the squares of tangents from it to circles A, B, C are x^2, y^2, z^2, respectively.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; A. H. HOLMES, Brunswick, Maine; L. E. NEWCOMB, Los Gatos, California; and J. SCHEFFER, A. M., Hagerstown, Md.

Let (u, v) be the co-ordinates of the point O;

$(x-m_1)^2+(y-n_1)^2=R_1^2$, the equation of the circle A,

$(x-m_2)^2+(y-n_2)^2=R_2^2$, the equation of the circle B, and

$(x-m_3)^2+(y-n_3)^2=R_3^2$, the equation of the circle C.

Then $(u-m_1)^2+(v-n_1)^2-R_1^2=T_1^2$, the square of the tangent from O to A;

$(u-m_2)^2+(v-n_2)^2-R_2^2=T_2^2$, the square of the tangent from O to B;

$(u-m_3)^2+(x-n_3)^2-R_3^2=T_3^2$, the square of the tangent from O to C.

$$\therefore 2(m_2-m_1)u+2(n_2-n_1)v+m_1^2+n_1^2-m_2^2-n_2^2+R_2^2-R_1^2=x^2 \ldots (1),$$

$$2(m_3-m_1)u+2(n_3-n_1)v+m_1^2+n_1^2-m_3^2-n_3^2+R_3^2-R_1^2=y^2 \ldots (2),$$

$$2(m_3-m_2)u+2(n_3-n_2)v+m_2^2+n_2^2-m_3^2-n_3^2+R_3^2-R_2^2=z^2 \ldots (3).$$

Adding (1) and (3) and subtracting (2), we have $x^2+z^2=y^2$ or $y^2-x^2=z^2$, the former being the equation of a circle with a variable radius y, and the latter the equation of an equilateral hyperbola with a variable semi-axes z.

If x, y, and z are constants, the locus is a point.

309. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

To find the equation of Brocard's Ellipse, the sides b and c of the triangle being the axes of co-ordinates.

Solution by the PROPOSER.

Let AB be the axis of x, AC that of y, then the equation of any ellipse touching the three sides of the triangle is of the form $D^2y^2+4Bxy+E^2x^2+4Dy+4Ex+4=0$, where

$$B=-\frac{8+4bD+4cE+bc\,DE}{2bc}.$$

The co-ordinates of the center of the ellipse are $-\frac{2D}{DE+2B}$ and $-\frac{2E}{DE+2B}$, or substituting the value of B, $\frac{Dbc}{2(2+bD+cE)}$ and $\frac{Ebc}{2(2+bD+cE)}$. The co-ordinates of the two Brocard Points O and O' are, respectively,

$$\frac{a^2b^2c}{a^2b^2+a^2c^2+b^2c^2},\ \frac{b^3c}{a^2b^2+a^2c^2+b^2c^2},\text{ and }\frac{b^2c^3}{a^2b^2+a^2c^2+b^2c^2},\ \frac{a^2c^2b}{a^2b^2+a^2c^2+b^2c^2}.$$

Therefore, the co-ordinates of the middle point of OO', that is, the center of the ellipse, are

$$\tfrac{1}{2}\cdot\frac{b^2c(a^2+c^2)}{a^2b^2+a^2c^2+b^2c^2}\text{ and }\tfrac{1}{2}\cdot\frac{bc^2(a^2+b^2)}{a^2b^2+a^2c^2+b^2c^2};$$

$$\therefore\ \frac{D}{2+bD+cE}=\frac{b(a^2+c^2)}{a^2b^2+a^2c^2+b^2c^2},\quad \frac{E}{2+bD+cE}=\frac{c(a^2+b^2)}{a^2b^2+a^2c^2+b^2c^2};$$

whence, $D=-\frac{2(a^2+c^2)}{bc^2}$, $E=-\frac{2(a^2+b^2)}{b^2c}$.

Substituting, we find for the required equation of the Brocard Ellipse,

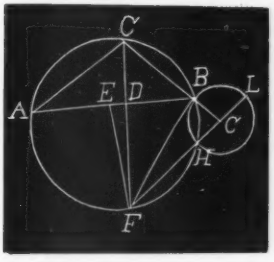
$$b^2(a^2+c^2)^2y^2+2bc[(a^2b^2+a^2c^2+b^2c^2)-a^4]xy+c^2(a^2+b^2)^2x^2$$
$$-2b^3c^2(a^2+c^2)y-2b^2c^3(a^2+b^2)x+b^4c^4=0.$$

Also solved by G. B. M. Zerr, who used trilinear co-ordinates.

310. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

Construct a plane triangle having given the base, the vertical angle, and the bisector of the vertical angle.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; J. SCHEFFER, A. M., Hagerstown, Md., and C. N. SCHMALL, A. B., 89 Columbia Street, New York.



Upon the given base AB construct a circle whose segment ACB shall contain the given vertical angle. Through E, the mid-point of AB, draw EF perpendicular to AB, meeting the circumference at F. Join FB, and perpendicular to FB draw BG equal to one half the given bisector of the vertical angle. With G as center and BG as radius describe the circle BHL, and draw FGL. With F as center, FL as radius, describe a circle cutting the given circle in C. Join FC, cutting AB in D. Then ABC is the triangle required.

In the triangles FCB and FBD, $\angle FCB = \angle FBA$, since arc AF = arc FB; also $\angle CFB$ is common, hence the triangles are similar, and $FC : FB = FB : FD$; but $FL(=FC) : FB = FB : FH$. Therefore $FH = FD$ and $HL = CD$.

Hence in the triangle ABC, AB is the given base, $\angle ACB$ the given vertical angle, and CD the given bisector, and the triangle is satisfied in every condition.

Also solved by L. E. Newcomb, and A. H. Holmes.

CALCULUS.

233. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Prove that $\int_0^\infty \frac{x^{a-1}dx}{1+2x\cos\theta+x^2} = \frac{\pi \sin(1-a)\theta}{\sin a\pi \sin\theta}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int_0^\infty \frac{a^{-1}dx}{1+2x\cos\theta+x^2} = \frac{1}{a}\int_0^\infty \frac{dx}{\sin^2\theta+(x+\cos\theta)^2}$$

$$= \frac{1}{a\sin\theta}\tan^{-1}\left(\frac{x+\cos\theta}{\sin\theta}\right)_0^\infty = \frac{\theta}{a\sin\theta}.$$

The problem giving the result stated is as follows:

$$\int_0^\infty \frac{x^{a-1}dx}{1+2x\cos\theta+x^2} = \frac{1-\sqrt{(-1)}\cot\theta}{2}\int_0^\infty \frac{x^{a-2}dx}{x+\cos\theta+\sqrt{(-1)}\sin\theta}$$

$$+\frac{1+\sqrt{(-1)}\cot\theta}{2}\int_0^\infty \frac{x^{a-2}dx}{x+\cos\theta-\sqrt{(-1)}\sin\theta} = P.$$

Let $x = y[\cos\theta \pm \sqrt{(-1)}\sin\theta]$, the plus sign for the first term, the minus sign for the second term.

$$\therefore P = \tfrac{1}{2}[1-\sqrt{(-1)}\cot\theta]\,[\cos(a-2)\theta+\sqrt{(-1)}\sin(a-2)\theta]\int_0^\infty \frac{y^{a-2}dy}{1+y}$$

$$+\tfrac{1}{2}[1+\sqrt{(-1)}\cot\theta]\,[\cos(a-2)\theta-\sqrt{(-1)}\sin(a-2)\theta]\int_0^\infty \frac{y^{a-2}dy}{1+y}.$$

$$\therefore P = [\cos(a-2)\theta+\sin(a-2)\theta\cot]\int_0^\infty \frac{y^{a-2}dy}{1+y}$$

$$= \frac{\sin(a-1)\theta}{\sin\theta}\int_0^\infty \frac{y^{a-2}dy}{1+y} = \frac{\sin(a-1)\theta}{\sin\theta}\cdot\frac{\pi}{\sin(a-1)\pi}.$$

$$\therefore P = \frac{\pi\sin(1-a)\theta}{\sin\theta\sin a\pi}.$$

This problem was incorrectly stated, the error being due to an oversight in reading proof. It is correctly stated above, the numerator being x^{a-1} instead of a^{-1}. ED. F.

234. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Find the first negative pedal of an ellipse semi-axes a, b, referred to origin as center, and show that its entire area is $\frac{\pi}{2}\left[\frac{(a^2+b^2)^2}{4ab}+ab\right]$.

Solution by the PROPOSER.

$r\cos(\theta-\phi)=p=\frac{b}{\sqrt{(1-e^2\cos^2\phi)}}$, is the equation of a straight line perpendicular to the radius vector at its extremity. The envelope of this line is the first negative pedal. Differentiating we get

$$\sin(\theta-\phi)\sqrt{(1-e^2\cos^2\theta)}+\frac{e^2\cos(\theta-\phi)\sin\phi\cos\phi}{\sqrt{(1-e^2\cos^2\phi)}}=0.$$

$$\cos(\theta-\phi)=\frac{b}{r\sqrt{(1-e^2\cos^2\phi)}},\quad \sin(\theta-\phi)=\frac{\sqrt{[r^2(1-e^2\cos^2\phi)-b^2]}}{\sqrt{(1-e^2\cos^2\phi)}}.$$

These values from the equation of tho line substituted in the derived equation gives us

$$r^2=\frac{b^2}{(1-e^2\cos^2\phi)^2}-\frac{e^2b^2(1-e^2)\cos^2\phi}{(1-e^2\cos^2\phi)^3}$$

as the pedal sought.

$$\therefore A=2a^2b^2\int_0^{\frac{1}{2}\pi}\left[\frac{a^2}{(a^2\sin^2\phi+b^2\cos^2\phi)^2}-\frac{b^2(a^2-b^2)\cos^2\phi}{(a^2\sin^2\phi+b^2\cos^2\phi)^3}\right]d\theta.$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(a^2\sin^2\phi+b^2\cos^2\phi)^2}=\frac{\pi(a^2+b^2)}{4a^3b^3}.$$

Differentiating this with respect to b we get

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\phi\,d\phi}{(a^2\sin^2\phi+b^2\cos^2\phi)^3}=\frac{\pi(3a^2+b^2)}{16a^3b^5}.$$

$$\therefore A=\frac{\pi a(a^2+b^2)}{2b}-\frac{\pi(a^2-b^2)(3a^2+b^2)}{8ab}=\frac{\pi}{8ab}(a^4+6a^2b^2+b^4)$$

$$=\frac{\pi}{2}\left[\frac{(a^2+b^2)^2}{4ab}+ab\right].$$

Also solved by J. Scheffer.

235. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

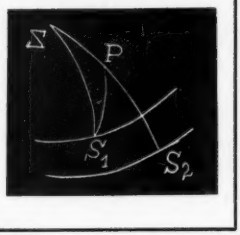
The latitude of a place and two circles parallel to the horizon being given, to determine the declination of a heavenly body whose apparent time of passage from one circle to the other shall be a minimum.

I. Solution by the PROPOSER.

Employing the usual notation, let Z be the zenith, P the celestial pole, S_1, S_2, the positions of a heavenly body on the parallel circles, the polar distances PS_1 and PS_2 being equal.

Let $\angle ZPS_1=\phi_1$, and $\angle ZPS_2=\phi_2$; and let $PS_1=PS_2=x$, arc $ZS_1=a_1$, $ZS_2=a_2$, latitude$=\lambda$, declination$=D$.

Then by the conditions of the problem the heavenly body is to pass along the arc S_1S_2 in the shortest possible time. In other words $\angle S_1PS_2$ is to be a minimum.



$$\therefore \frac{d}{dx}S_1PS_2=\frac{d}{dx}(\phi_2-\phi_1)=0,$$

$$\text{but } \frac{d\phi_1}{dx}=-\frac{\cot S_1}{\sin x} \text{ and } \frac{d\phi_2}{dx}=-\frac{\cot S_2}{\sin x}.$$

$$\therefore \frac{\cot S_1}{\sin x}=\frac{\cot S_2}{\sin x}. \quad \therefore S_1=S_2;$$

$$\text{but } \cos S_1=\frac{\sin\lambda-\cos a_1\cos x}{\sin a_1\sin x} \text{ and } \cos S_2=\frac{\sin\lambda-\cos a_2\cos x}{\sin a_2\sin x}.$$

$$\therefore \frac{\sin\lambda-\cos a_1\cos x}{\sin a_1}=\frac{\sin\lambda-\cos a_2\cos x}{\sin a_2}.$$

$$\text{Whence } \cos x=\frac{\cos\frac{1}{2}(a_2+a_1)}{\cos\frac{1}{2}(a_2-a_1)}.\sin\lambda.$$

The declination being the complement of the polar distance we have, therefore,

$$\sin D=\frac{\cos\frac{1}{2}(a_2+a_1)}{\cos\frac{1}{2}(a_2-a_1)}\sin\lambda.$$

In the special case where $a_1=\frac{1}{2}\pi$, and $a_2=\frac{1}{2}\pi+2\delta$, this expression is reduced to $\sin D=-\tan\delta\sin\lambda$. This formula may have important applications. Thus, take for instance, the case of the sun, and supposing 2δ to be its angular depression below the horizon when twilight begins in the morning or ends in the evening, we can calculate the time of shortest twilight at a given place by means of the relation $-\sin D=\tan\delta\tan\lambda$, the negative sign indicating that when the latitude is north the declination will be south, and *vice versa*.

II. Solution by W. D. LAMBERT, Washington, D. C.

This is a slight generalization of Bernoulli's classic problem of the shortest twilight. The expressions that occur in the analytic solution are long and cumbrous, but present no essential difficulty. An outline of the analytic solution is given in Hutton's *Course of Mathematics*, Vol. II, page 385 (12th ed.). The following synthetic solution may be new to many readers.

Suppose for definiteness the body, S, to be west of the meridian. Let P be the pole of the celestial sphere, Z the zenith when the body is in the upper circle of zenith distance $z=ZS$. After the lapse of a period t, the point of the sphere once occupied by Z has moved to Z', and the body S is now in S' at a zenith distance $z+2a=ZS'$. Let ϕ denote the latitude, $=90°-PZ$, and δ the declination $=90°-PS=90°-PS'$. $ZPZ'=SPS'=t$.

The arc ZZ' is found from the isosceles triangle ZPZ' and is given by $\sin\frac{1}{2}ZZ'=\cos\phi\sin\frac{1}{2}t$, so that ZZ' increases with t, and will have a minimum value when t has one. From the triangle $ZZ'S'$,

$$\cos ZZ'=\cos z\cos(z+2a)+\sin z\sin(z+2a)\cos ZS'Z'\ldots(1),$$

z and $z+2a$ are given. The only variable on the right is $ZS'Z'$, and since $\sin z$ and $\sin(z+2a)$ are always positive, by increasing $\cos ZS'Z'$ we increase $\cos ZZ'$ and thereby diminish arc ZZ'. ZZ', and with it t, will be a minimum when $ZS'Z'=0$, and then by (1), $ZZ'=2a$, which implies that Z' falls on $S'Z$. From the triangle ZPS',

$$\sin\delta=\cos(z+2a)\sin\phi+\cos\phi\sin(z+2a)\cos PZS'\ldots(2),$$

but from the triangle ZPZ', $\cos PZS'=\tan a\tan\phi$, so that (2) may be reduced to

$$\sin\delta=\frac{\sin\phi\cos(z+a)}{\cos a}\ldots(3),$$

which is the required declination.

The corresponding time is given by

$$\sin\frac{t}{2}=\frac{\sin a}{\cos\phi}\ldots(4).$$

For the problem of shortest twilight, $Z=90°$, and $2a$ is usually taken as 18°. Equations (2) and (3) then show that the central parts of this country, of latitude say 40°, have their shortest twilights when the sun is 5° 51′ south of the equator, or about March 6, and October 8. These twilights last 1 hour, 34 minutes.

It is easily seen from a figure that the parallactic angles at the beginning and end of the minimum period, namely PSZ and $PS'Z$, are equal, and the azimuths PZS and PZS' are supplementary. Some such symmetry as this regard to the prime vertical might be anticipated, for the rate of descent in altitude of a a body is a maximum when on the prime vertical.

Excellent solutions of this problem were received from G. B. M. Zerr, J. Scheffer, and G. W. Greenwood.

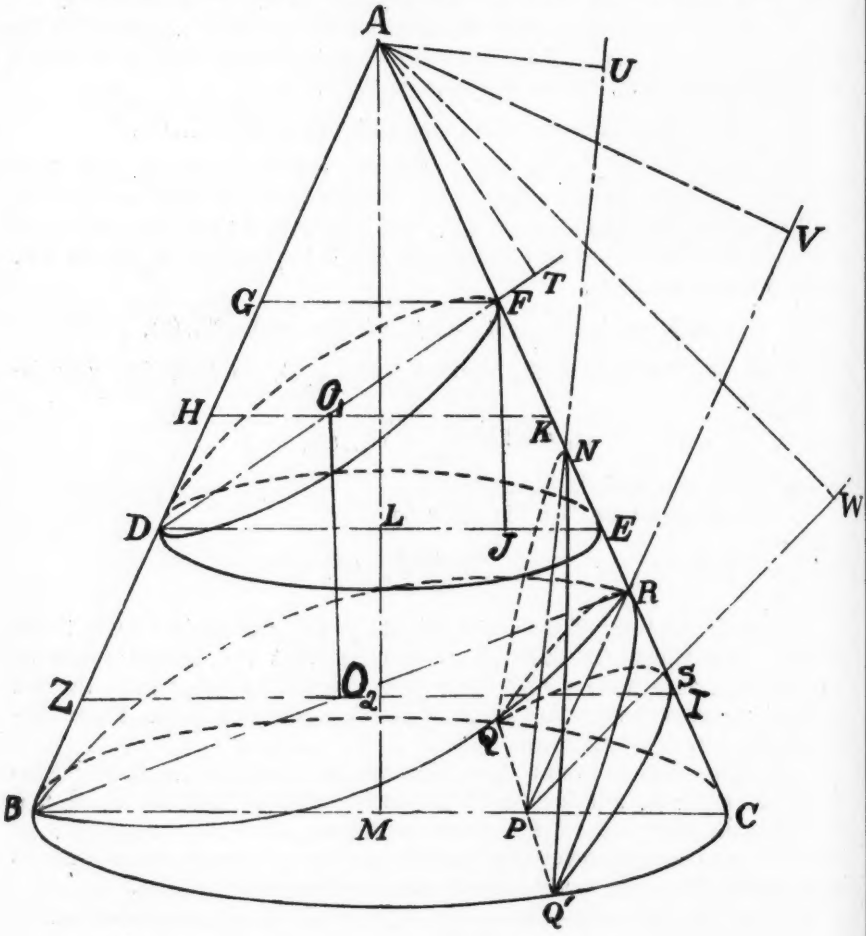
MECHANICS.

196. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

From a uniform, solid right circular cone two planes cut off a portion such that the sections are similar ellipses with co-planar axes (not parallel). The centers of the elliptic faces are O_1, O_2, and the center of gravity of the solid is G. GX parallel to O_1O_2 cuts the axis of the cone in X. Find GX/O_1O_2 in terms of the ratio of the major axes of the ellipses.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the cone; BR, DF the major axes of the required ellipses. For calculations we will only consider the cone ADE and the plane DF.



Let $x^2+z^2=n^2(c-y)^2$ be the cone, $x=my-R$ the plane, where $R=DL$, $c=AL$, $h=FJ$, $r=\frac{1}{2}GF$, $n=\tan LAE=(R-r)/h$, $m=\tan FDE=(R+r)/h$, $c=Rh/(R-r)$. Let $my-R=x_1$, $n(c-y)=x_2$, $(x+nc)/n=y_1$, $(x+R)/m=y_2$, $(nc-x)/n=y_3$.

$$\therefore v=2\int_0^h \int_{-x_2}^{x_1} \sqrt{[n^2(c-y)^2-x^2]}\,dydx+2\int_h^c \int_{-x_2}^{x_2} \sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$=\int_0^h \left\{ n^2(c-y)^2\left[\tfrac{1}{2}\pi+\sin^{-1}\left(\frac{my-R}{n(c-y)}\right)\right]\right.$$

$$\left.+(my-R)\sqrt{[n^2(c-y)^2-(my-R)^2}\right\}dy+\pi n^2\int_h^c (c-y)^2dy=\frac{\frac{1}{3}\pi Rrh\sqrt{Rr}}{R-r}.$$

$$\therefore v=\tfrac{1}{3}\pi\ (AD.AF)^{\frac{1}{2}}\sin^2\beta\cos\beta, \text{ where } \angle LAE=\beta.$$

$$\bar{x}v=2\int_0^h \int_{-x_2}^{x_1} x\sqrt{]n^2(c-y)^2-x^2]}\,dydx+2\int_h^c \int_{-x_2}^{x_2} x\sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$=-\tfrac{2}{3}\int_0^h [n^2(c-y)^2-(my-R)^2]^{\frac{3}{2}}\,dy$$

$$=-\frac{2}{3(m^2-n^2)^{\frac{3}{2}}}\int_{-u_1}^{u_1}[(ncm-Rn)^2-u^2]^{\frac{3}{2}}\,du$$

where $u=(m^2-n^2)y+n^2c-Rm$, $u_1=2rR/h$.

$$\therefore \bar{x}v=-\tfrac{1}{8}\pi Rrh\sqrt{(Rr)}=-\tfrac{1}{8}\pi(AD-AF)(AD.AF)^{\frac{1}{2}}\sin^2\beta\cos\beta.$$

$$\bar{y}v=2\int_0^h \int_{-x_2}^{x_1} y\sqrt{[n^2(c-y)^2-x^2]}\,dydx+2\int_h^c \int_{-x_2}^{x_2} y\sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$=2c\int_0^h \int_{-x_2}^{x_1} \sqrt{[n^2(c-y)^2-x^2]}\,dydx+2c\int_h^c \int_{-x_2}^{x_2} \sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$-2\int_{-r}^{r}\int_{y_3}^{h}(c-y)\sqrt{[n^2(c-y)^2-x^2]}\,dxdy$$

$$-2\int_{-R}^{-r}\int_{y_2}^{y_1}(c-y)\sqrt{[n^2(c-y)^2-x^2]}\,dxdy$$

$$-2\int_{-r}^{r}\int_{h}^{y_2}(c-y)\sqrt{[n^2(c-y)^2-x^2]}\,dx\,dy$$

$$=\frac{\frac{1}{3}\pi cRrh\sqrt{Rr}}{R-r}-\frac{2}{3n^2m^3}\int_{-R}^{r}[n^2(cm-R-x)^2-m^2x^2]^{\frac{3}{2}}\,dx=\frac{\frac{1}{3}\pi cRrh\sqrt{Rr}}{R-r}$$

$$-\frac{2}{3n^2m^3(m^2-n^2)^{\frac{1}{2}}}\int_{-u_2}^{u_2}[(ncm^2-nRm)^2-x^2]^{\frac{3}{2}}\,du,$$

where $u=(m^2-n^2)x^2+cmn^2-Rn^2$, $u_2=\dfrac{2Rr(R+r)}{h^2}$.

$$\therefore \bar{y}v=\frac{\pi Rrh^2\sqrt{(Rr)}(5R-3r)}{24(R-r)^2}=\frac{\pi}{24}(5AD-3AF)(AD.AF)^{\frac{1}{2}}\sin^2\beta\cos^2\beta.$$

$\bar{x}$, $\bar{y}$ give the center of gravity of the cone ADF. The center of gravity G of the portion $FDBRF$ is given as follows:

$$\bar{x}_1=\frac{3[(AD-AF)(AD.AF)^{\frac{1}{2}}-(AB-AR)(AB.AR)^{\frac{1}{2}}]\sin\beta}{8[(AB.AR)^{\frac{1}{2}}-(AD.AF)^{\frac{1}{2}}]}=a.$$

$$\bar{y}_1=\frac{[(5AB-3AR)(AB.AR)^{\frac{1}{2}}-(5AD-3AF)(AD.AF)^{\frac{1}{2}}.]\cos\beta}{8[(AB.AR)^{\frac{1}{2}}-(AD.AF)^{\frac{1}{2}}]}=b.$$

Now $DF=\sqrt{(AD^2+AF^2-2AD.AF\cos 2\beta)}$,
$BR=\sqrt{(AB^2+AR^2-2AB.AR\cos 2\beta)}$.
Coordinates of O_1 are, $[AB-\frac{1}{2}(AD+AF)]\cos\beta$, and, $\frac{1}{2}(AD-AF)\sin\beta$,
Coordinates of O_2 are, $\frac{1}{2}(AB-AR)\cos\beta$, and, $\frac{1}{2}(AB-AR)\sin\beta$.
$\therefore O_1O_2=\frac{1}{2}][(AB-AD)^2+(AR-AF)^2+2(AB-AD)(AR-AF)\cos 2\beta]^{\frac{1}{2}}$.

GX makes with the axis of abscissas an angle whose tangent is

$$\left(\frac{AD+AF-AB-AR}{AB+AF-AR-AD}\right)\cot\beta=\mu.$$

$\therefore y-b=\mu(x-a)$ is the equation to GX. This meets the axis of ordinates in $x=0$, $y=b-\mu a$.

$\therefore GX=a\sqrt{(1+\mu^2)}$

$$=\frac{a\sqrt{[(AB-AD)^2+(AR-AF)+2(AB-AD)(AR-AF)\cos 2\beta}}{(AB+AF-AR-AD)\sin\beta}$$

$$=\frac{2aO_1O_2}{(AB+AF-AR-AD)\sin\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{2a}{(AB+AF-AR-AD)\sin\beta}$$

$$=\frac{6[(AD-AF)(AD.AF)^{\frac{1}{2}}-(AB-AR)(AB.AR)^{\frac{1}{2}}]}{8(AB+AF-AR-AD)[(AB.AR)^{\frac{1}{2}}-(AD.AF)^{\frac{1}{2}}]}.$$

Let $\angle FDE=\theta$, $\angle RBC=\phi$, semi-minor axis of ellipse $DF=p$, of ellipse $BR=q$. Then

$$p=\surd(HO_1.O_1K)=\surd(AD.AF)\sin\beta, \qquad q=\surd(ZO_2.O_2I)=\surd(AB.AR)\sin\beta,$$

$$AD-AF=FE=\frac{DF\sin\theta}{\cos\beta}, \qquad AB-AR=RC=\frac{BR\sin\phi}{\cos\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{6(p^3DF\sin\theta-q^3BR\sin\phi)}{8(BR\sin\phi-DF\sin\theta)(q^3-p^3)}.$$

Let $DF=2P$, $BR=2Q$. Then

$$p=\frac{P\surd[\cos(\theta+\beta)\cos(\theta-\beta)]}{\cos\beta}=\frac{P\surd[\cos^2\theta-\sin^2\beta]}{\cos\beta}, \quad q=\frac{Q\surd[\cos^2\theta-\sin^2\beta]}{\cos\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{6[P^4\sin\theta(\cos^2\theta-\sin^2\beta)^{\frac{3}{2}}-Q^4\sin\phi(\cos^2\phi-\sin^2\beta)^{\frac{3}{2}}]}{8\{[Q\sin\phi-P\sin\theta][Q^3(\cos^2\phi-\sin^2\beta)^{\frac{3}{2}}-P^3(\cos^2\theta-\sin^2\beta)^{\frac{3}{2}}]\}}.$$

If the planes of the ellipses are parallel, $\theta=\phi$, and we get

$$\frac{GX}{O_1O_2}=\frac{6(P^4-Q^4)}{8(Q-P)(Q^3-P^3)}=\frac{6(P^2+Q^2)(P+Q)}{P^3-Q^3}.$$

MISCELLANEOUS.

165. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that $\tan^{-1}\frac{n}{n+1}+\tan^{-1}\frac{1}{2n+1}=\frac{1}{4}\pi$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; A. H. HOLMES, Brunswick, Me.; J. EDWARD SANDERS, Reinersville, O.; FRANK M. DRYZER, A. B., Knoxville, Tenn.; and PROF. J. W. NICHOLSON, State University, La.

Let $\alpha=\tan^{-1}\frac{n}{n+1}$ and $\beta=\tan^{-1}\frac{1}{2n+1}$.

Then $\tan\alpha=\frac{n}{n+1}$, $\tan\beta=\frac{1}{2n+1}$, and $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{\frac{n}{n+1}+\frac{1}{2n+1}}{1-\frac{n}{(n+1)(2n+1)}}=\frac{2n^2+2n+1}{2n^2+2n+1}=1.$$

$\therefore \alpha+\beta=\frac{1}{4}\pi$.

Also solved by G. W. Greenwood.

166. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

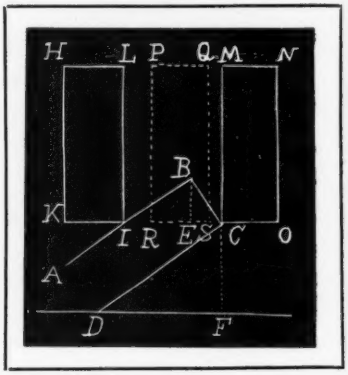
Several equal rectangular boxes are placed in a row with uniform intervals between the boxes and a passageway along one side of the row. Find the least width of the passageway permitting a box to be removed from the row without moving adjacent boxes. This problem arose during the construction of a room for storage batteries.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let *HLIK*, *MNOC* be two of the boxes, with the box *PQSR* occupying the space between them in the position *ABCD*. Let $KH=MC=AB=a$, the length of the box; $KI=CO=BC=b$, the width of the box; $IR=SC=c$, the interval between boxes.

When the box is in the position *ABCD* it can be taken out without removing the adjacent box or boxes.

Then *CF* is the width of the passageway. $CF=CD\sin CDF=a\sin\theta$.



$$IC=\frac{BC}{\cos\theta}=\frac{b}{\cos\theta}=b+2c.$$

$$\therefore\cos\theta=\frac{b}{b+2c}. \quad \therefore CF=\frac{2a}{b+2c}\sqrt{[c(b+c)]}.$$

$$\therefore \text{Width}=\frac{2a}{b+2c}\sqrt{[c(b+c)]}.$$

If the height of the box is less than the width, and *IC* wide enough to turn the box on its side, then write $b=$height and $c=\frac{1}{2}(IC-\text{height})$.

In the above $a>b+2c$, otherwise the passageway would be equal to the width of the box plus a few inches for room.

Also solved by A. H. Holmes.

167. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

If possible, arrange 43 objects, say the numbers 0, 1, 2, ..., in 43 sets of seven each such that every pair of objects lies in one and only one set of seven. It will then be true that two sets of seven have in common one and only one object.

Discussion by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

The following process of building successive sets of seven each leads to the conclusion that the problem is impossible. Let any set of seven be 0, 42, 41, ..., 37. As no two of this set can occur again together, each of these elements must next be combined with six sets of six elements each, these six elements being chosen from the numbers 36, 35, ..., 1. It will be shown that the twelve sets below may be taken as twelve of the desired sets additional to the above.

42, 36, 35, 34, 33, 32, 31	41, 36, 30, 24, 18, 12, 6
42, 30, 29, 28, 27, 26, 25	41, 35, 29, 23, 17, 11, 5
42, 24, 23, 22, 21, 20, 19 (*A*),	41, 34, 28, 22, 16, 10, 4 (*B*).
42, 18, 17, 16, 15, 14, 13	41, 33, 27, 21, 15, 9, 3
42, 12, 11, 10, 9, 8, 7	41, 32, 26, 20, 14, 8, 2
42, 6, 5, 4, 3, 2, 1	41, 31, 25, 19, 13, 7, 1

In (A), 42 is any one of the first set, and the remainder of the rows must be all different both within themselves and from each other, as written.

Proceeding to (B), after choosing say, 41, from the first set, each row must contain one and only one element from each row of (A). Since the elements in the several rows of (A) are permutable, it is allowable to take the columns in (A) for the remainders of the rows of (B). For convenience call the result of omitting 42 from (A) the *square*. Further discussion is confined to this square. As the square rotated about its principal diagonal appears in (B), any *new* six must be chosen from it as one would obtain a term from a determinant. But (A) and (B) admit interchanges of both columns and rows, without losing even this reciprocal relation, so that any new six may become the principal diagonal. Hence 36, 29, 22, 15, 8, 1, may be taken as any new six, and is to be coupled with one of the elements 0, 40, 39, 38, 37, to make the fourteenth set of seven. So far 36 has been used three times, hence it must appear in four more sets of six, and the same is true of 29, 22, 15, 8, 1, while none of these six elements may appear in the same set. Thus twenty-four sets of six are accounted for in addition to the above fourteen sets of seven. The remaining sets of six, five in number, will come from the square, with the diagonal omitted.

The square possesses two transformations which are of use in reducing the labor of computing sets of six. The first is a translation along the diagonal, with provision for the edges, and is

$$\begin{matrix}(1,\ 8,\ 15,\ 22,\ 29,\ 36) & (2,\ 9,\ 16,\ 23,\ 30,\ 31) & (3,\ 10,\ 17,\ 24,\ 25,\ 32) \\ (4,\ 11,\ 18,\ 19,\ 26,\ 33) & (5,\ 12,\ 13,\ 20,\ 27,\ 34) & (6,\ 7,\ 14,\ 21,\ 28,\ 35)\end{matrix} \quad (C).$$

The second is the rotation about the diagonal. By a method to be shown, it is possible to find all the consistent sets of four sixes which contain 36. Now (C) leaves invariant the fourteen sets of seven already found, but gives by successive applications to the sets containing 36, those containing 29, then 22, etc. There are fifty-six sets of four sixes for 36, but six of them are invariant under the rotation, coupled with (42, 41), the fourteen sets of seven being also invariant. The remaining fifty sets are, by the rotation, reducible to twenty-five, so that thirty-one sets of four sixes for 36 are to be used. Of course there must be used with these, the entire fifty-six sets for 29, 22, etc.

In computing the sets for 36 the outline is this:

$$\begin{matrix}36,\ 28,\ (19,\ 20,\ 21,\ 23) \\ 36,\ 27,\ (19,\ 20,\ 23) \\ 36,\ 26,\ (19,\ 21,\ 23) \\ 36,\ 25,\ (20,\ 21,\ 23)\end{matrix} \quad (D), \qquad \begin{matrix}36,\ 28,\ 19,\ (14,\ 17) \\ 36,\ 27,\ 20,\ (13,\ 16,\ 17) \\ 36,\ 26,\ 21,\ (13,\ 16,\ 17) \\ 36,\ 25,\ 23,\ (14,\ 16)\end{matrix} \quad (E),$$

in which the parentheses in (D) indicate those elements of the third line of the square which are available for the third element of the respective rows in (D). This gives eleven partial results from (D), of which the first is

(*E*). In this way the total of fifty-six sets of four sixes for 36 is obtained.

The remainder of the work begins with the comparison of such of the thirty-one sets for 36 with each of the fifty-six sets for 29. Although in upwards of forty cases one obtains consistent sets of eight sixes, all of these cases fail at the trial with the sets for 22, showing that the problem is impossible.

PROBLEMS FOR SOLUTION.

ALGEBRA.

283. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Solve $w+x+y+z=4a$, $w^2+x^2+y^2+z^2=4a^2+4b^2$, $w^3+x^3+y^3+z^3=4a^3+12ab^2$, $w^4+x^4+y^4+z^4=4a^4+4b^4+4c^4+24a^2b^2$.

GEOMETRY.

316. Proposed by J. STEWART GIBSON, Department of Physics, Wadleigh High School, New York City.

Determine the locus of the vertices of parabolas described by particles thrown off from the circumference of a uniformly revolving wheel.

CALCULUS.

239. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

Of all triangles inscribed in a circle, find that which has the greatest perimeter.

MECHANICS.

202. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Three equal, uniform, similar rods AB, BC, CD, freely jointed at B and C, are hung from a point by two equal strings attached at A and D. Find the position of equilibrium.

MISCELLANEOUS.

171. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $\underset{x=a}{\text{Lim.}} \frac{\phi(x)}{\psi(x)}=\lambda$, show $\underset{x=a}{\text{Lim.}} \left[\frac{\lambda}{\phi(x)}-\frac{1}{\psi(x)}\right]=\frac{\lambda\psi''(a)-\phi''(a)}{2\phi'(a)\psi'(a)}$.

ERRATA.

Page 97, line 10. Vol. XIII, for $x=y=w=$ etc., read $x=x_1=x_2=$ etc.
Page 98, line 1, for $G+D+U$ read $G+D+U+B$, B taken from table.
Page 97, in table add .008 to each number from 34 to 43 inclusive.

5

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THE COMPLETE PAPPUS HEXAGON.*

By DR. C. C. GROVE, Hamilton College, Clinton, N. Y.

1. Propositions 138 and 139, Book VII of Pappus Alexandrinus,† as translated by F. Hultsch reads thus:

"Iam his demonstratis ostendendum erit, si parallelae sint $\alpha\beta$ $\gamma\delta$, et in eas incidant quaedam rectae $\alpha\delta$ $\alpha\varsigma$ $\beta\gamma$ $\beta\varsigma$, quarum $\alpha\delta$ $\beta\gamma$ concurrant in μ, et a quovis rectae $\alpha\beta$ puncto inter α et β sumpto ducantur $\epsilon\gamma$ $\epsilon\delta$, quarum $\epsilon\gamma$ cum $\alpha\varsigma$ concurrant in η et $\varepsilon\delta$ cum $\beta\varsigma$ in κ, rectam esse quae per $\eta\mu\kappa$ transit."

"At ne sint parallelae $\alpha\beta$ $\gamma\delta$, sed convergant in puncto ν; dico rursus rectam esse quae per $\eta\mu\kappa$ transit."

These theorems Pappus proved by proportion, the equal ratios being respectively between two lines and between the rectangles of two pairs of lines.

Salmon‡ has the same theorems stated thus:

"If ABC are three points of one line and $A'B'C'$ are three points of another line, then the intersections $BC'/B'C$, $CA'/C'A$, $AB'/A'B$ lie on a line."

Different other writers have Pappus's theorem in some wording. The most important mention of the simple case is by Rudolph Boeger,§ who gives it as a simple form, free from the idea of projective relations, of "*Das Sechseck in der Geometrie der Lage.*"

Some other papers directly or indirectly presenting perspective triangles are simply noted:

H. Schroeter: Math. Annalen 2:553—562.

J. Vályi: Archiv der Math. und Physik, 1882, Bd. 70, ss. 105—110; 1884. 2. R. II. T., ss. 230—234.

Rosanes: Ueber Dreiecke in persp. Lage. Math. Ann. 2:549.

Hess: Beitraege z. Theorie d. mehrfach persp. Dreiecke. Ibid. 28: 167.

*Read before the American Mathematical Society, Chicago Section, April 30, 1907.

†Pappi Mathematicae Collectiones a Federico Commandino Urbinate, in Latin, in U. S Cong. Library. Also, by F. Hultsch in three volumes (Greek and Latin), Berolini, 1877.

‡Conic Sections:—6th Ed., §268, p. 246, Ex. 1.

§*Sechseck und Involution*, Hamburg Mitteilungen, Bd. III., 9, Feb. 1899, s. 387.